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The head-on interaction of two solitary waves of unequal amplitude

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In this paper we consider the interaction of two unequal solitary waves travelling in opposite directions. The interaction equations are derived from the perturbation expansion of the Euler equations, which lead to the Boussinesq equation at lowest order. These equations are recast to obtain two weakly coupled KdV equations which are solved by the method of inverse scattering. We show that the amplitude of each solitary wave after interaction is reduced. This change in the amplitudes is shown to be fifth order in ϵ, the order of magnitude of the amplitude of the waves. This is in agreement with the change of amplitude produced by the interaction of two equal waves which arises in the reflection of a wave by a vertical wall.

1. Introduction

The equations for the propagation of water waves of small amplitude and long wavelength were first derived by Boussinesq (1872) and Korteweg & De Vries (1895). The Boussinesq equation allows waves to propagate in opposite directions while the Korteweg–de Vries (KdV) equation describes only the unidirectional propagation of waves. Both equations have solitary wave solutions which are functions of a single phase variable. Byatt-Smith (1971) showed that while the Boussinesq equation admitted solutions that consisted of two such solitary waves of arbitrary amplitude travelling in opposite directions, the equation itself does not separate. It becomes necessary to add to the superposition of the two solitary waves a smaller-order interaction term in order that the travelling wave terms separate into functions of their respective phase variables. For the Boussinesq equation itself the criterion of separability is sufficient to determine the interaction term to any desired order. Moreover the interaction term can be expressed as a uniformly valid expansion, this result presumably being a consequence of the fact that the Boussinesq equation is integrable. Miles (1977) showed, in the more general case of obliquely interacting waves, that this interaction could at lowest order be expressed as a phase shift and a transient term.

Su & Mirie (1980, 1982) completed this procedure to the third order for waves travelling in opposite directions. However, it becomes apparent that the expansion procedure, which is valid for a single wave, is not uniformly valid for all times for the interaction terms. This is because at higher order the predicted phase shift is not constant and results in a deformed wave after interaction, which does not satisfy the steady equation for a single wave. The distorted wave eventually separates into a steady wave and a dispersive tail.

In a recent paper Byatt-Smith (1988) corrected the expansion procedure to allow for slowly deforming waves by deriving coupled perturbed KdV equations for each wave. In that paper he solved the problem of a solitary wave reflected by a vertical wall and showed that the reflected wave was reduced in amplitude. This reduction was shown to be fifth order in ϵ, the order of magnitude of the amplitude of the wave. The result of this change of amplitude is that a dispersive tail of order ϵ^3 is produced behind the reflected wave. The paper by Byatt-Smith extended the result of Su & Mirie (1980) who obtained the dispersive tail but could not theoretically predict any change in amplitude.

The case of a reflected solitary wave is equivalent to two equal waves travelling in opposite directions, assuming that symmetry breaking perturbations are not unstable. In this paper we consider the case of two solitary waves of different amplitudes initially far apart which are travelling towards each other. As in Byatt-Smith (1988) we derive, from Euler equations for fluid flow, interaction equations which are perturbations of the KdV equations. It is shown that the loss of amplitude of each wave during interaction is again of fifth order and that the dispersive tail behind each wave after interaction is third order. We also show that at this order there is no transfer of energy between the two waves, so that after interaction the sum of the energy in the reduced solitary wave and in its dispersive tail is the same as the energy in the original wave.

2. Basic equations

We consider unsteady, two-dimensional irrotational motion of a fluid. The motion is assumed to be such that all disturbances tend to zero at infinity where the depth, h, is uniform. It will be convenient to choose units so that

$$h = g = 1, \tag{2.1}$$

where g is the acceleration due to gravity.

Let (x, y) be horizontal and vertical coordinates, t the time, η the free-surface displacement and φ the velocity potential. The boundary-value problem is then described by

$$\varphi_{xx} + \varphi_{yy} = 0 \quad (0 < y < 1+\eta), \tag{2.2}$$

$$\varphi_y = 0 \quad (y = 0), \tag{2.3}$$

$$\eta_t + \varphi_x \eta_x - \varphi_y = 0 \quad (y = 1+\eta), \tag{2.4}$$

and

$$\eta + \varphi_t + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2 = 0. \tag{2.5}$$

Following Miles (1977), Su & Mirie (1980) or Byatt-Smith (1987*a, b*, 1988) we look for a solution of the form

$$\varphi(x, y, t) = \sum_0^\infty (-\mathrm{D}^2)^n \frac{\Phi(x,t) y^{2n}}{(2n)!}, \tag{2.6}$$

where $\mathrm{D} \equiv \partial/\partial x$.

In terms of η and $W(x, t) \equiv \partial\Phi/\partial x$, (2.4) and the x-derivative of (2.5) can be written as

$$\eta_t + W_x + \mathrm{D}\left\{\eta W + \sum_1^\infty (-1)^n \frac{(1+\eta)^{2n+1}}{(2n+1)!} \mathrm{D}^{2n} W\right\} = 0, \tag{2.7}$$

and

$$W_t+\eta_x+\mathrm{D}\left\{\tfrac{1}{2}W^2+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\eta)^{2n}}{2n!}\right.$$
$$\left.\times\left[\mathrm{D}^{2n-1}\partial_t W+\tfrac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\mathrm{D}^m W\mathrm{D}^{2n-m}W\right]\right\}=0, \quad (2.8)$$

where $\partial_t \equiv \partial/\partial t$.

By adding and subtracting these two equations they may be rewritten as

$$(\partial_t \pm \mathrm{D})(W\pm\eta)+\mathrm{D}F_{\pm}=0, \quad (2.9)$$

where

$$F_{\pm}=\tfrac{1}{2}W^2\pm W\eta+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\eta)^{2n}}{2n!}$$
$$\times\left[\partial_t\mathrm{D}^{2n-1}W\pm\frac{1+\eta}{2n+1}\mathrm{D}^{2n-1}W+\tfrac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\mathrm{D}^m W\mathrm{D}^{2n-m}W\right]. \quad (2.10)$$

We now consider two solitary waves initially far apart, of small but finite amplitude travelling towards each other. In the absence of any interaction terms each solitary wave will be a function of a single phase variable and we introduce new coordinates

$$\xi_1=\epsilon^{\frac{1}{2}}k_1(x-c_1 t), \quad \xi_2=\epsilon^{\frac{1}{2}}k_2(x+c_2 t), \quad (2.11)$$

where $0<\epsilon\ll 1$ is a small parameter representing the order of magnitude of the wave amplitudes which are given by ϵa_i $(i=1,2)$.

The respective wavenumber and phase velocities are given by $\epsilon^{\frac{1}{2}}k_i$ and c_i. To obtain equations describing the two waves and their interaction we introduce the notation

$$\partial_1=\partial/\partial\xi_1, \quad \partial_2=\partial/\partial\xi_2, \quad (2.12)$$

and the change of dependent variable

$$\alpha=\tfrac{1}{2}\epsilon^{-1}(W+\eta), \quad \beta=\tfrac{1}{2}\epsilon^{-1}(\eta-W). \quad (2.13)$$

In terms of these variables (2.2) becomes

$$2\epsilon(c_2+c_1)\,k_2\,\partial_2\,\alpha+(k_1\,\partial_1+k_2\,\partial_2)\tilde{F}_+=0, \quad (2.14)$$

and

$$2\epsilon(c_1+c_2)\,k_1\,\partial_1\beta+(k_1\,\partial_1+k_2\,\partial_2)\tilde{F}_-=0, \quad (2.15)$$

where now†

$$\tilde{F}_{\pm}=-2\left(\begin{Bmatrix}c_1\\c_2\end{Bmatrix}-1\right)\epsilon\begin{Bmatrix}\alpha\\\beta\end{Bmatrix}+\tfrac{1}{2}\epsilon^2(\alpha-\beta)^2\pm\epsilon^2(\alpha^2-\beta^2)$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\epsilon(\alpha+\beta))^{2n}}{2n!}\left[\epsilon^{n+1}(c_2\,k_2\,\partial_2-c_1\,k_1\,\partial_1)\,(k_1\,\partial_1+k_2\,\partial_2)^{2n-1}(\alpha-\beta)\right.$$
$$\pm\frac{(1+\epsilon(\alpha+\beta))}{2n+1}\epsilon^{n+1}(k_1\,\partial_1+k_2\,\partial_2)^{2n}(\alpha-\beta)$$
$$\left.+\tfrac{1}{2}\epsilon^{n+2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}(k_1\,\partial_1+k_2\,\partial_2)^m(\alpha-\beta)\,(k_1\,\partial_1+k_2\,\partial_2)^{2n-m}(\alpha-\beta)\right]. \quad (2.16)$$

Following Byatt-Smith (1988) we propose to model the interaction by deriving equations for the unsteady travelling wave part of the solutions. This will take the

† The minus sign before the first term of (2.16) replaces the $\mp$ sign which is in error in the corresponding equation, (3.6), in Byatt-Smith (1988).

form of approximate KdV equations which can be analysed by the method of inverse scattering. To do this we retain only the terms that lead to singular behaviour as $t \to +\infty$. (See Byatt-Smith 1988 for further details of this procedure.) A preliminary requirement in deriving the interaction equations, see for example Su & Mirie (1980), is that

$$c_i = 1+\tfrac{1}{2}\epsilon a_i + O(\epsilon^2), \quad k_i^2 = 3a_i + O(\epsilon). \tag{2.17}$$

The interaction equations are then

$$4k_2\,\partial_2\alpha + \epsilon k_1(a_1\,\partial_1^3\alpha + 3\alpha\,\partial_1\,\alpha - a_1\,\partial_1\,\alpha) = \epsilon^2 k_1(9\alpha - 4a_1)\,\partial_1\,\alpha\beta, \tag{2.18}$$

and

$$4k_1\,\partial_1\,\beta + \epsilon k_2(a_2\,\partial_2^3\beta + 3\beta\,\partial_2\,\beta - a_2\,\partial_2\,\beta) = \epsilon^2 k_2(9\beta - 4a_2)\,\partial_2\,\beta\alpha. \tag{2.19}$$

We transform (2.11) and (2.12) to approximate KdV equations by writing

$$\alpha = -2a_1\,u, \beta = -2a_2\,v, \quad \tau_1 = k_2\,a_2\,\epsilon\xi_1/(2k_1) \text{ and } \tau_2 = k_1\,a_1\,\epsilon\xi_2/(2k_2). \tag{2.20}$$

In terms of these variables (2.11) and (2.12) are

$$\frac{\partial u}{\partial \tau_2} = 3u\,\partial_1\,u - \tfrac{1}{2}\partial_1^3\,u + \tfrac{1}{2}\partial_1\,u + \epsilon a_2(18u+4)\,\partial_1\,uv, \tag{2.21}$$

and

$$\frac{\partial v}{\partial \tau_1} = 3v\,\partial_2\,v - \tfrac{1}{2}\partial_2^3 v + \tfrac{1}{2}\partial_2 v + \epsilon a_1(18v+4)\,\partial_2\,vu. \tag{2.22}$$

The transformation given by (2.20) has been chosen so that (2.21) and (2.22) coincide as far as possible with those derived by Byatt-Smith (1988) (cf. (5.1) and (5.2) of that paper).

3. The interaction of two solitary waves

Equations (2.21) and (2.22) are both perturbations of the KdV equation connected via the interaction terms. These equations may be solved using the method of perturbed inverse scattering developed by Karpman & Maslov (1977*a–c*), Keener & Mclaughlin (1977) and Kaup & Newell (1978) and used by Byatt-Smith (1988), which should be consulted for further details.

The general solitary wave solution of the unperturbed equation

$$\tilde{u}_{\tau_2} = 3\tilde{u}\,\partial_1\,\tilde{u} - \tfrac{1}{2}\partial_1^3\,\tilde{u} + \tfrac{1}{2}\partial_1\,\tilde{u}, \tag{3.1}$$

is given in terms of the single variable

$$\theta_1 = \kappa_1\,\xi_1 - \tfrac{1}{2}(\kappa_1^3 - \kappa_1)\,\tau_2, \tag{3.2}$$

and takes the form

$$\tilde{u}_0(\theta) = -\tfrac{1}{2}\kappa_1^2\,\mathrm{sech}^2\,\tfrac{1}{2}\theta_1. \tag{3.3}$$

In our case we have defined coordinates in which the unperturbed wave before interaction is stationary so that $\kappa_1 = 1$. We denote this solution by u_0 so that

$$u_0(\xi_1) = -\tfrac{1}{2}\,\mathrm{sech}^2\,(\tfrac{1}{2}\xi_1). \tag{3.4}$$

To obtain the change of amplitude we must expand u to second order and proceed by writing

$$u = u_0(\xi_1) + \epsilon^2 u_1(\xi_1, \tau_2, \epsilon) + \epsilon^4 u_2(\xi_1, \tau_2, \epsilon) + o(\epsilon^4). \tag{3.5}$$

The term u_1 represents the dispersive wave train and satisfies the linearized equation

$$\frac{\partial u_1}{\partial \tau_2} = 3\partial_1(u_1\,u_0) - \tfrac{1}{2}\partial_1^3\,u_1 + \tfrac{1}{2}\partial_1\,u_1 + \epsilon^{-1}a_2(18u_0+4)\,\partial_1\,u_0\,v_0, \tag{3.6}$$

Since, to this order, there is no change of amplitude due to interaction, we may obtain the solution of (3.6) from the solution $U_\mu(\xi_1, \tau_2)$ of the homogeneous equation. This is expressed as

$$U_\mu(\xi_1, \tau_2) = F_1(\xi_1, \mathrm{k})\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\mu\tau_2 + \mathrm{i}k\xi_1},$$

where
$$\mu = -(k+k^3), \tag{3.8}$$

and
$$F_1 = \mathrm{i}k(k^2-1) - 4ik u_0 - 2\partial_1 u_0 + 2k^2\,\partial_1 u_0/u_0. \tag{3.9}$$

The solution u_1 is then expressed as a convolution integral in the form

$$\begin{aligned} u_1(\zeta_1, \tau_2) &= -\epsilon^{-1} a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\tau_2} v_0(2k_2\tau_0/(k_1 a_1 \epsilon))\,\bar{u}_0(k)\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\mu(\tau_2-\tau_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\tau_0\,\mathrm{d}k, \\ &= -\tfrac{1}{2}k_1 k_2^{-1} a_1 a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\xi_2} v_0(\xi_0)\,\bar{u}_0(k)\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu_1\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\xi_0\,\mathrm{d}k, \end{aligned} \tag{3.10}$$

where
$$\bar{u}_0(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi_1} u_0(\xi_1)\,\mathrm{d}\xi_1 = -\frac{k}{\sinh \pi k}, \tag{3.11}$$

and
$$\mu_1 = k_1 a_1 \mu/k_2. \tag{3.12}$$

The first approximation to (2.22) is obtained in a similar fashion and

$$v_1 = \tfrac{1}{2}k_2 k_1^{-1} a_1 a_2 \int_{-\infty}^{\infty}\int_{\xi_1}^{\infty} u_0(\xi_0)\,\bar{v}_0(k)\,F_2(\xi_2, k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\bar{\mu}_2(\xi_1-\zeta_0)+\mathrm{i}k\xi_2}\,\mathrm{d}\xi_0\,\mathrm{d}k. \tag{3.13}$$

We proceed to the next approximation by introducing (3.5) into (2.14) and linearizing with respect to u_2. This gives the equation

$$\begin{aligned} \frac{\partial u_2}{\partial \tau_2} = {} & 3\partial_1(u_2 u_0) - \tfrac{1}{2}\partial_1^3 u_2 + \tfrac{1}{2}\partial_1 u_2 \\ & + a_2\epsilon^{-1}\{(18u_0+4)\,\partial_1 u_0 v_1 + \partial_1((18u_0+4)\,u_1)\,v_0\} + 3\partial_1 u_0^2. \end{aligned} \tag{3.14}$$

In order to obtain a uniformly valid solution we must first account for the change of amplitude of the solitary wave. This arises because the inhomogeneous term in (3.14) is not orthogonal to u_0 when integrated over $(-\infty, \infty)$. It is this integral that governs the rate of change of the amplitude. Byatt-Smith (1987*a*, *b*) shows that the application of the method of perturbed inverse scattering gives rise to the equation

$$\begin{aligned} \frac{\mathrm{d}\kappa_1}{\mathrm{d}\tau_2} &= \tfrac{1}{8}\epsilon^4 \int_{-\infty}^{\infty} u_0 a_0 \epsilon^{-1}\{(18_0+4)\,\partial_1 u_0 v_1 + \partial_1((18u_0+4)\,u_1)\,v_0\}\,\mathrm{d}\xi_1 \\ &= \tfrac{1}{8}a_2\epsilon^3 \int_{-\infty}^{\infty} \{u_0(18u_0+4)\,\partial_1 u_0 v_1 - (18u_0+4)\,u_1\,\partial_1 u_0 v_0\}\,\mathrm{d}\xi_1, \end{aligned} \tag{3.15}$$

the contributions from the term $3\partial_1(u_0^2)$ being of smaller order. The total change in κ_1 during interaction is then given by

$$\begin{aligned} [\kappa_1] &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa_1}{\mathrm{d}\tau_2}\mathrm{d}\tau_2 = \tfrac{1}{2}k_2^{-1}k_1 a_1 \epsilon \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa_1}{\mathrm{d}\tau_2}\mathrm{d}\xi_2 \\ &= \tfrac{1}{32}\epsilon^4(a_2^2 a_1^2 I_1 + a_2 a_1^3 I_2), \end{aligned} \tag{3.16}$$

where

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\xi_0=\xi_1}^{\infty} u_0(\xi_1)(9u_0(\xi_1)+2)\,\partial_1 u_0(\xi_1)\,u_0(\xi_0)\,\bar{v}_0(k)\,F_2(\xi_2,k) \times \mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu_2\epsilon(\xi_1-\xi_0)+\mathrm{i}k\xi_2}\,\mathrm{d}\xi_0\,\mathrm{d}k\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \tag{3.17}$$

and

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_0=\xi} (9u_0(\xi_1)+2)\,\partial_1 u_0(\xi_1)\,v_0(\xi_2)\,v_0(\xi_0)\,\bar{u}_0(k)\,F_1(\xi_1,k) \times \mathrm{e}^{-\frac{1}{4}\mathrm{i}\tilde{\mu}_1\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\xi_0\,\mathrm{d}k\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \tag{3.18}$$

Byatt-Smith (1988) evaluates these integrals to obtain

$$I_1 = 0 \quad \text{to} \quad \mathrm{O}(\epsilon), \tag{3.19}$$

and

$$I_2 = -40/21 \quad \text{to} \quad O(\epsilon). \tag{3.20}$$

Thus

$$[\kappa_1] = -\tfrac{5}{42}a_1^3 a_2\,\epsilon^4 + O(\epsilon^5). \tag{3.21}$$

Using the definition of the individual waves (2.6) and the change of variable (2.13) the unscaled amplitude $\tilde{a}_1$ is given by

$$\tilde{a}_1 = \epsilon a_1 \kappa_1^2, \tag{3.22}$$

so that

$$\begin{aligned}[\tilde{a}_1] = 2\epsilon a_1[\kappa_1] &= -\tfrac{5}{21}\epsilon^5 a_1^4 a_2 + O(\epsilon^6) \\ &= -\tfrac{5}{21}\tilde{a}_1^4\,\tilde{a}_2 + O(\epsilon^6).\end{aligned} \tag{3.23}$$

The importance of (3.19) is slightly obscured by the symmetry in the case of two equal waves but is easily interpreted in the case of unequal waves. We now show using energy arguments that the term involving the integral I_1 in (3.16) represents the exchange of energy from one wave to the other while the term involving I_2 represents the transfer of energy from one wave to its own dispersive tail. Thus (3.19) implies that during interaction no energy is transferred between the two waves at this order and that the result of the interaction is to redistribute the energy of the solitary wave into a smaller wave and a dispersive tail.

To leading order the energy of a solitary wave of amplitude $\tilde{a}$ is given by

$$\begin{aligned}E_{1\mathrm{s}} &= \int_{-\infty}^{\infty}\eta^2\,\mathrm{d}x = 4\epsilon^2 a_1^2\int_{-\infty}^{\infty}u_0^2\,\epsilon^{-\frac{1}{2}}k_1^{-1}\,\mathrm{d}\xi = \text{const}\times(\epsilon a_1)^{\frac{3}{2}} \\ &= E_0\,\tilde{a}_1^{\frac{3}{2}}.\end{aligned} \tag{3.24}$$

Using a modified version of (3.23) for the case $I_1 \neq 0$ the change of energy of the solitary wave is

$$[E_{1\mathrm{s}}] = \tfrac{3}{2}E_0\,\tilde{a}_1^{\frac{1}{2}}[\tilde{a}_1] = -\tfrac{5}{14}E_0\,\tilde{a}_1^{\frac{1}{2}}[\tilde{a}_2^2\,\tilde{a}_3^3\,I_1/I_2 + \tilde{a}_1^4\,\tilde{a}_2]. \tag{3.25}$$

On the assumption that the solitary wave and its dispersive tail are well separated the energy in the dispersive tail is

$$E_{1_\mathrm{t}} = \int_{-\infty}^{\infty}(4\epsilon^3 a_1 u_1)^2\,\frac{\mathrm{d}\xi_1}{\epsilon^{\frac{1}{2}}k_1^{\frac{1}{2}}} = \text{const}\times\tilde{a}_1^{\frac{9}{2}}\tilde{a}_2. \tag{3.26}$$

This integral is easily associated with the second term in (3.25) by consideration of the symmetric case of two equal waves. Thus the first term represents the energy transfer from wave to wave which by (3.19) is zero.

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